EJERCICIO 1 (21:38 del video)

Dado:

$$\underline{\boldsymbol{A}} = \left(\boldsymbol{b^0}, \vec{\boldsymbol{b}}\right) \sin \boldsymbol{kx}$$

Calcular los campos eléctrico y magnético

$$\boldsymbol{kx} = \boldsymbol{\omega t} - \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}}$$

$$\vec{\boldsymbol{r}} = \boldsymbol{x}\,\hat{\boldsymbol{x}} + \boldsymbol{y}\,\hat{\boldsymbol{y}} + \boldsymbol{z}\,\hat{\boldsymbol{z}}$$

$$\vec{\boldsymbol{k}} = \boldsymbol{k_x}\hat{\boldsymbol{x}} + \boldsymbol{k_y}\hat{\boldsymbol{y}} + \boldsymbol{k_z}\hat{\boldsymbol{z}}$$

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\,\mathrm{V} - \frac{\boldsymbol{\partial}\,\vec{\mathbf{A}}}{\boldsymbol{\partial}\,\mathbf{t}}$$

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$$

$$V = b^0 \sin kx$$

$$\vec{\nabla}\,\mathrm{V} = \vec{\nabla}\,(\,b^0 \sin kx) = b^0\vec{\nabla}\left(\sin\left(\omega t - \vec{k}\cdot\vec{r}\right)\right) = b^0\left(-\vec{k}\right)\cos kx$$

$$\frac{\partial\,\vec{\mathrm{A}}}{\partial\,\mathrm{t}} = \frac{\partial\left(\vec{b}\sin kx\right)}{\partial\,\mathrm{t}} = \vec{b}\;\omega\cos kx$$

$$\vec{E} = -\vec{\nabla}\,\mathrm{V} - \frac{\partial\,\vec{\mathrm{A}}}{\partial\,\mathrm{t}} = -b^0\left(-\vec{k}\right)\cos kx - \vec{b}\;\omega\cos kx$$

$$\vec{E} = b^0\,\vec{k}\cos kx - \vec{b}\;\omega\cos kx$$

$$\boxed{\vec{\boldsymbol{E}} = \left(\boldsymbol{b^0}\,\vec{\boldsymbol{k}} - \vec{\boldsymbol{b}}\,\boldsymbol{\omega}\right)\cos \boldsymbol{kx}}$$

$$\vec{B} = \vec{\nabla}\times\vec{\mathrm{A}} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ b_x\sin kx & b_y\sin kx & b_z\sin kx \end{pmatrix} = \begin{pmatrix} \partial_y(b_z\sin kx) - \partial_z\left(b_y\sin kx\right) \\ -\partial_x(b_z\sin kx) + \partial_z(b_x\sin kx) \\ \partial_x\left(b_y\sin kx\right) - \partial_y(b_x\sin kx) \end{pmatrix}$$

$$\vec{B} = \vec{\nabla}\times\vec{\mathrm{A}} = \begin{pmatrix} b_z\left(-k_y\right)\cos kx - b_y(-k_z)\cos kx \\ -b_z(-k_x)\cos kx + b_x(-k_z)\cos kx \\ b_y\,(-k_x)\cos kx - b_x\left(-k_y\right)\cos kx \end{pmatrix} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ -k_x & -k_y & -k_z \\ b_x & b_y & b_z \end{pmatrix}\cos kx$$

$$\boxed{\vec{\boldsymbol{B}} = -\left(\vec{\boldsymbol{k}}\times\vec{\boldsymbol{b}}\right)\cos \boldsymbol{kx}}$$

EJERCICIO 2 (40:05 del video)

Dado

$$\boldsymbol{\underline{A}} = \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b^1} \\ \boldsymbol{b^2} \\ \boldsymbol{\beta} \end{pmatrix} \mathbf{sin}\, \boldsymbol{kx}$$

expresar el cuadrivector en función de $\boldsymbol{e^{-ikx}}$ $\boldsymbol{y}$ $\boldsymbol{e^{ikx}}$

$$\mathbf{sin}\, \boldsymbol{kx} = \frac{\boldsymbol{e^{ikx} - e^{-ikx}}}{\boldsymbol{2i}}$$

$$\underline{A} = \begin{pmatrix} \beta \\ b^1 \\ b^2 \\ \beta \end{pmatrix} \sin kx = \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \sin kx + b^1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sin kx + b^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sin kx + \beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \sin kx$$

$$\underline{\varepsilon}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \underline{\varepsilon}_1 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \underline{\varepsilon}_2 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \underline{\varepsilon}_3 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} \beta \\ b^1 \\ b^2 \\ \beta \end{pmatrix} \sin kx = \beta \underline{\varepsilon}_0 \sin kx + b^1 \underline{\varepsilon}_1 \sin kx + b^2 \underline{\varepsilon}_2 \sin kx + \beta \underline{\varepsilon}_3 \sin kx$$

$$\underline{A} = \beta \underline{\varepsilon}_0 \frac{e^{ikx} - e^{-ik}}{2i} + b^1 \underline{\varepsilon}_1 \frac{e^{ikx} - e^{-ikx}}{2i} + b^2 \underline{\varepsilon}_2 \frac{e^{ikx} - e^{-ikx}}{2i} + \beta \underline{\varepsilon}_3 \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$\underline{A} = \left(\frac{\beta}{2i} \underline{\varepsilon}_0 + \frac{b^1}{2i} \underline{\varepsilon}_1 + \frac{b^2}{2i} \underline{\varepsilon}_2 + \frac{\beta}{2i} \underline{\varepsilon}_3 \right) e^{ikx} - \left(\frac{\beta}{2i} \underline{\varepsilon}_0 + \frac{b^1}{2i} \underline{\varepsilon}_1 + \frac{b^2}{2i} \underline{\varepsilon}_2 + \frac{\beta}{2i} \underline{\varepsilon}_3 \right) e^{-ikx}$$

$$c^0 = c^3 \equiv -\frac{\beta}{2i}; c^1 \equiv -\frac{b^1}{2i}; c^2 \equiv -\frac{b^2}{2i}$$

$$\underline{A} = \left(-c^0 \underline{\varepsilon}_0 - c^1 \underline{\varepsilon}_1 - c^2 \underline{\varepsilon}_2 - c^3 \underline{\varepsilon}_3\right) e^{ikx} - \left(-c^0 \underline{\varepsilon}_0 - c^1 \underline{\varepsilon}_1 - c^2 \underline{\varepsilon}_2 - c^3 \underline{\varepsilon}_3\right) e^{-ikx}$$

$c^r = -(c^r)^*$ donde $(c^r)^*$ es el complejo conjugado

$$\underline{A} = \left((c^0)^* \underline{\varepsilon}_0 + (c^1)^* \underline{\varepsilon}_1 + (c^2)^* \underline{\varepsilon}_2 + (c^3)^* \underline{\varepsilon}_3\right) e^{ikx} + \left(c^0 \underline{\varepsilon}_0 + c^1 \underline{\varepsilon}_1 + c^2 \underline{\varepsilon}_2 + c^3 \underline{\varepsilon}_3\right) e^{-ikx}$$

$$\underline{A} = \sum_{r=0}^{3} (c^r)^* \underline{\varepsilon}_r e^{ikx} + c^r \underline{\varepsilon}_r e^{-ikx}$$

Como $\underline{\varepsilon}_r$ son reales, $\underline{\varepsilon}_r = \left(\underline{\varepsilon}_r\right)^*$

$$\boxed{\boldsymbol{\underline{A}} = \sum_{\boldsymbol{r=0}}^{\boldsymbol{3}} (\boldsymbol{c^r})^* \left(\boldsymbol{\underline{\varepsilon}_r}\right)^* \boldsymbol{e^{ikx}} + \boldsymbol{c^r} \boldsymbol{\underline{\varepsilon}_r} \boldsymbol{e^{-ikx}}}$$

EJERCICIO 3 (47:30 del video)

Dado:

$\underline{\boldsymbol{A}}^{(2)} = \left(\boldsymbol{b}^{0}, \vec{\boldsymbol{b}}\right) \cos \boldsymbol{kx}$

Calcular los campos eléctrico y magnético

$$V = b^{0} \cos kx$$

$$\vec{\nabla}\,\mathrm{V} = \vec{\nabla}\,(\,b^{0} \cos kx) = b^{0}\,\vec{\nabla}\left(\cos\left(\omega t - \vec{k}\cdot\vec{r}\right)\right) = b^{0}\left(-\vec{k}\right)(-\sin kx)$$

$$\frac{\partial\,\vec{\mathrm{A}}^{(2)}}{\partial\,\mathrm{t}} = \frac{\partial\,\left(\vec{b}\cos kx\right)}{\partial\,\mathrm{t}} = -\vec{b}\;\omega \sin kx$$

$$\vec{E}^{(2)} = -\,\vec{\nabla}\,\mathrm{V} - \frac{\partial\vec{\mathrm{A}}^{(2)}}{\partial\,\mathrm{t}} = -b^{0}\vec{k}\sin kx + \vec{b}\;\omega \sin kx$$

$$\vec{E} = -b^{0}\;\vec{k}\sin kx + \vec{b}\;\omega\sin kx$$

$$\boxed{\vec{\boldsymbol{E}} = \left(-\boldsymbol{b}^{0}\,\vec{\boldsymbol{k}} + \vec{\boldsymbol{b}}\;\boldsymbol{\omega}\right)\sin \boldsymbol{kx}}$$

$$\vec{B} = \vec{\nabla}\times\vec{\mathrm{A}} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ b_x\cos kx & b_y\cos kx & b_z\cos kx \end{pmatrix} = \begin{pmatrix} \partial_y(b_z\cos kx) - \partial_z\left(b_y\cos kx\right) \\ -\partial_x(b_z\cos kx) + \partial_z(b_x\cos kx) \\ \partial_x\left(b_y\cos kx\right) - \partial_y(b_x\cos kx) \end{pmatrix}$$

$$\vec{B} = \vec{\nabla}\times\vec{\mathrm{A}} = \begin{pmatrix} b_z\left(-k_y\right)(-\sin kx) - b_y(-k_z)(-\sin kx) \\ -b_z(-k_x)(-\sin kx) + b_x(-k_z)(-\sin kx) \\ b_y(-k_x)(-\sin kx) - b_x\left(-k_y\right)(-\sin kx) \end{pmatrix}$$

$$= \begin{pmatrix} b_z\left(k_y\right)(\sin kx) - b_y(k_z)(\sin kx) \\ -b_z(k_x)(\sin kx) + b_x(k_z)(\sin kx) \\ b_y(k_x)(\sin kx) - b_x\left(k_y\right)(\sin kx) \end{pmatrix}\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & k_y & k_z \\ b_x & b_y & b_z \end{pmatrix}\sin kx$$

$$\boxed{\vec{\boldsymbol{B}} = \left(\vec{\boldsymbol{k}}\times\vec{\boldsymbol{b}}\right)\sin \boldsymbol{kx}}$$

EJERCICIO 4 (53:43 del video)

Dado

$$\boldsymbol{\underline{A}} = \begin{pmatrix} \boldsymbol{\alpha^0} \\ \boldsymbol{-E_0/\omega} \\ \boldsymbol{0} \\ \boldsymbol{\alpha^0} \end{pmatrix} \boldsymbol{\sin kx} + \begin{pmatrix} \boldsymbol{\beta^0} \\ \boldsymbol{0} \\ \boldsymbol{E_0/\omega} \\ \boldsymbol{\beta^0} \end{pmatrix} \boldsymbol{\cos kx}$$

expresar el cuadrivector como

$$\boldsymbol{\underline{A}} = \sum_{r=0}^{3} (\boldsymbol{c^r})^* \left(\boldsymbol{\underline{\varepsilon}_r}\right)^* \boldsymbol{e^{ikx}} + \boldsymbol{c^r \underline{\varepsilon}_r e^{-ikx}}$$

$$\boldsymbol{\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}};\quad \boldsymbol{\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}}$$

$$\underline{A} = \begin{pmatrix} \alpha^0 \\ -E_0/\omega \\ 0 \\ \alpha^0 \end{pmatrix} \frac{e^{ikx} - e^{-ikx}}{2i} + \begin{pmatrix} \beta^0 \\ 0 \\ E_0/\omega \\ \beta^0 \end{pmatrix} \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\underline{A} = \left\{ \frac{1}{2i} \begin{pmatrix} \alpha^0 \\ -E_0/\omega \\ 0 \\ \alpha^0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \beta^0 \\ 0 \\ E_0/\omega \\ \beta^0 \end{pmatrix} \right\} e^{ikx} + \left\{ -\frac{1}{2i} \begin{pmatrix} \alpha^0 \\ -E_0/\omega \\ 0 \\ \alpha^0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \beta^0 \\ 0 \\ E_0/\omega \\ \beta^0 \end{pmatrix} \right\} e^{-i}$$

$$\underline{A} = \begin{pmatrix} \dfrac{\alpha^0}{2i} + \dfrac{\beta^0}{2} \\ -\dfrac{E_0}{2i\omega} \\ \dfrac{E_0}{2\omega} \\ \dfrac{\alpha^0}{2i} + \dfrac{\beta^0}{2} \end{pmatrix} e^{ikx} + \begin{pmatrix} -\dfrac{\alpha^0}{2i} + \dfrac{\beta^0}{2} \\ \dfrac{E_0}{2i\omega} \\ \dfrac{E_0}{2\omega} \\ -\dfrac{\alpha^0}{2i} + \dfrac{\beta^0}{2} \end{pmatrix} e^{-ikx}$$

Si hacemos $\underline{A} = \underline{v}_+ e^{ikx} + \underline{v}_- e^{-ik}$

Se ve que $\underline{v}_- = \left(\underline{v}_+\right)^* = c^r \underline{\varepsilon}_r = \begin{pmatrix} -\frac{\alpha^0}{2i} + \frac{\beta^0}{2} \\ \frac{E_0}{2i\omega} \\ \frac{E_0}{2\omega} \\ -\frac{\alpha^0}{2i} + \frac{\beta^0}{2} \end{pmatrix}$

$$c^r \underline{\varepsilon}_r = \left(-\frac{\alpha^0}{2i} + \frac{\beta^0}{2}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{E_0}{2i\omega} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{iE_0}{2\omega} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}$$

Descomponemos el primer cuadrivector en dos vectores ortonormales.

$$c^r \underline{\varepsilon}_r = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{iE_0}{2\omega}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix} + \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

El segundo cuadrivector también podríamos descomponerlo en dos vectores ortonormales, pero da justo el cuadrivector $\underline{\varepsilon}_{+1}$. Primero lo normalizamos.

$$c^r \underline{\varepsilon}_r = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{iE_0}{2\omega}\sqrt{2}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix} + \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Para completar la base hace falta un vector que se denomina $\underline{\varepsilon}_{-1}$

$$c^r \underline{\varepsilon}_r = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{iE_0}{\omega\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix}\right\} + 0\underline{\varepsilon}_{-1} + \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Y que debe ser ortonormal a los otros vectores de la base

$$\left(\underline{\varepsilon}_{+1}\right)^* \cdot \underline{\varepsilon}_{-1} = 0$$

$$(\varepsilon^1{}_+)^*\varepsilon^1{}_- + (\varepsilon^2{}_+)^*\varepsilon^2{}_- = (-1)\varepsilon^1{}_- + i\,\varepsilon^2{}_- = -\varepsilon^1{}_- + i\,\varepsilon^2{}_- = 0$$

Es decir que si $\varepsilon^1{}_- = 1 \;\Rightarrow\; \varepsilon^2{}_- = -i$

$$c^r \underline{\varepsilon}_r = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{iE_0}{\omega\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix}\right\} + 0\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}\right\} + \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\boxed{\boldsymbol{c^0 = c^3 = \left(\frac{\alpha^0}{2} i + \frac{\beta^0}{2}\right);\; c^{+1} = \frac{iE_0}{\omega\sqrt{2}}\;;\; c^{-1} = 0}}$$

$$\boxed{\boldsymbol{\underline{\varepsilon}_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\; \underline{\varepsilon}_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix};\; \underline{\varepsilon}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix};\; \underline{\varepsilon}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}}$$